

CYCLIC BRANCHED COVERS OF ALTERNATING KNOTS AND L -SPACES

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ABSTRACT. For any alternating knot, it is known that the double branched cover of the 3-sphere branched over the knot is an L -space. We show that the three-fold cyclic branched cover is also an L -space for any genus one alternating knot.

1. INTRODUCTION

An L -space M is a rational homology 3-sphere whose Heegaard Floer homology $\widehat{HF}(M)$ is a free abelian group of rank equal to $|H_1(M; \mathbb{Z})|$ ([10]). The most typical examples of L -spaces are lens spaces. In recent years, it is recognized that L -spaces form an important class of 3-manifolds. For example, see [2, 10].

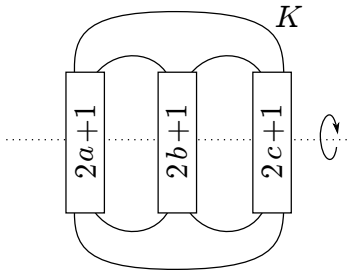
We consider the problem when cyclic branched covers of the 3-sphere branched over a knot or link is an L -space. Toward this direction, Ozsváth and Szabó [11] first showed that the double branched cover of any non-split alternating link (more generally, quasi-alternating link) is an L -space. Peters [13] verified that for a genus one, 2-bridge knot $C[2m, 2n]$ ($m, n > 0$) in Conway's notation, the d -fold cyclic branched cover is an L -space for any $d \geq 2$, and that for $C[2m, -2n]$ ($m, n > 0$), so is the 3-fold cyclic branched cover. For the latter, the same conclusion still holds for the cases $d = 4$ ([14]) and $d = 5$ ([8]), but it would be false for sufficiently large d ([9, 14]).

In this paper, we restrict ourselves to alternating knots. As mentioned above, the double branched cover of any alternating knot is an L -space. Then, is the 3-fold cyclic branched cover an L -space? The answer is positive for genus one, 2-bridge knots. However, it is negative, in general. Let $\Sigma_d(K)$ denote the d -fold cyclic branched cover of the 3-sphere branched over a knot K . By Baldwin [1], if K is the trefoil, then $\Sigma_d(K)$ is an L -space if and only if $d \leq 5$. This implies that if K is a $(2, m)$ -torus knot with $m \geq 7$, then $\Sigma_3(K)$ is not an L -space. For, $\Sigma_3(K)$ is homeomorphic to the m -fold cyclic branched cover of the trefoil. These torus knots

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FIGURE 1. A pretzel knot $K = P(2a + 1, 2b + 1, 2c + 1)$

are alternating, but have genus greater than one. Thus, we will examine the case where alternating knots have genus one.

Theorem 1.1. *Let K be a 3-strand pretzel knot $P(2a + 1, 2b + 1, 2c + 1)$, where $a, b, c > 0$. Then $\Sigma_3(K)$ is an L -space.*

This immediately implies the following.

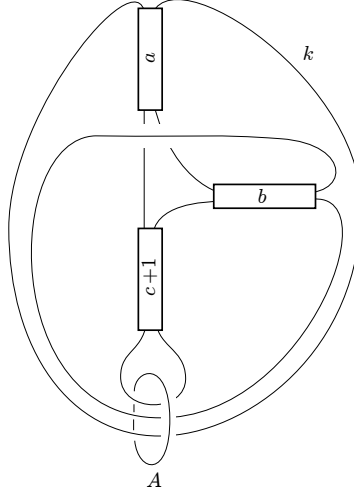
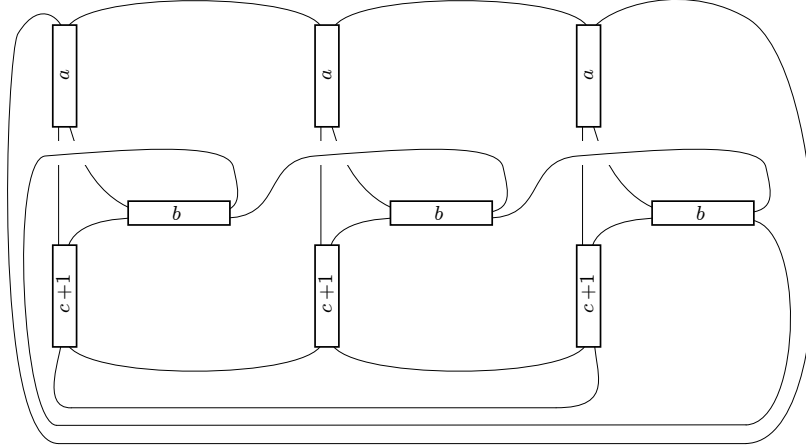
Corollary 1.2. *Let K be a genus one, alternating knot. Then $\Sigma_3(K)$ is an L -space.*

Proof. Suppose that K is a genus one, alternating knot. By [3, Lemma 3.1] (see also [12]), K is either a 2-bridge knot or a 3-strand pretzel knot $P(\ell, m, n)$ where ℓ, m, n have the same sign. For a genus one, 2-bridge knot, Peters [13] shows that $\Sigma_3(K)$ is an L -space. If $K = P(\ell, m, n)$, then ℓ, m, n are odd by [6]. Thus Theorem 1.1 gives the conclusion. \square

Hence, the rest of paper is devoted to prove Theorem 1.1. In Section 2, we describe a link \mathcal{L} whose double branched cover is homeomorphic to $\Sigma_3(K)$ for $K = P(2a + 1, 2b + 1, 2c + 1)$. Then Theorem 1.1 immediately follows from Theorem 2.2, which claims that the link \mathcal{L} is quasi-alternating. Section 3 describes how to calculate determinants of links through Goeritz matrices. In Section 4, we first argue the case where $a = 1$. Section 5 completes the proof of Theorem 2.2 by using an inductive argument. The last section contains some remarks.

2. QUASI-ALTERNATING LINKS

Let K be a pretzel knot $P(2a + 1, 2b + 1, 2c + 1)$ with $a, b, c > 0$, as illustrated in Figure 1. Here, each rectangular box consists of vertically right-handed half-twists of indicated number. This knot has cyclic period two such that its axis is drawn as the horizontal line. By taking the quotient of this action, the images of K and the axis give a link $k \cup A$ in Figure 2. The central two boxes consist of vertical twists, and the right box consists of horizontal twists. Note that each component of this link is unknotted. Moreover, it is easy to see that two components are interchangeable.

FIGURE 2. The link $k \cup A$ FIGURE 3. The link \mathcal{L}

Proposition 2.1. *Let \mathcal{L} be the link obtained as the lift of A in $\Sigma_3(k)$, which is the 3-sphere. Then $\Sigma_2(\mathcal{L})$ is homeomorphic to $\Sigma_3(K)$.*

Proof. Let M be the $\mathbb{Z}_3 \oplus \mathbb{Z}_2$ branched cover of $k \cup A$, corresponding to the map $H_1(S^3 - k \cup A) \rightarrow \mathbb{Z}_3 \oplus \mathbb{Z}_2$ sending positively oriented meridians of k and A to $(1, 0)$ and $(0, 1)$, respectively. Then M is homeomorphic to $\Sigma_2(\mathcal{L})$ and $\Sigma_3(K)$. \square

After exchanging the position of k and A in Figure 2, we still have the same diagram. Consider $\Sigma_3(k)$. Then the link \mathcal{L} is as illustrated in Figure 3.

We recall that the notion of quasi-alternating links [11]. The set of *quasi-alternating links* QA is the smallest set of links satisfying the following.

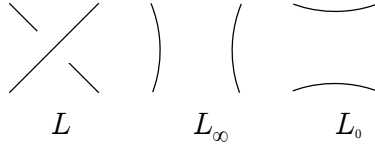


FIGURE 4. Resolutions

- The trivial knot belongs to QA.
- If a link L has a digram with crossing c such that both of two links L_∞ and L_0 obtained by smoothing c as in Figure 4 belong to QA, and $\det L = \det L_\infty + \det L_0$, then L belongs to QA.

As noted in Section 1, the double branched cover of a quasi-alternating link is an L -space, and any non-split alternating link is quasi-alternating (see [11]).

Theorem 2.2. *The link \mathcal{L} is quasi-alternating. Hence, $\Sigma_2(\mathcal{L})$ is an L -space.*

The proof of this theorem is split into Sections 4 and 5.

Proof of Theorem 1.1. By Proposition 2.1, $\Sigma_3(K)$ is homeomorphic to $\Sigma_2(\mathcal{L})$, which is an L -space by Theorem 2.2. \square

3. DETERMINANT

To show that the link \mathcal{L} is quasi-alternating, it is necessary to calculate the determinant of \mathcal{L} and those of various links arisen from \mathcal{L} by resolutions. These calculations are done through Goeritz matrices (see [4]).

First, consider the checkerboard coloring of the diagram of \mathcal{L} as in Figure 3. The unbounded region is white, and this region will be ignored. The vertical a right-handed half-twists at the upper left yield the white regions $\alpha_1, \alpha_4, \dots, \alpha_{3a-2}$ numbered from the top. Similarly, the white regions $\alpha_2, \alpha_5, \dots, \alpha_{3a-1}$ and $\alpha_3, \alpha_6, \dots, \alpha_{3a}$ appear at the upper center and the upper right. The three white regions just above horizontal b twists are numbered $\alpha_{3a+1}, \alpha_{3a+2}, \alpha_{3a+3}$ from the left. Finally, the white regions $\alpha_{3a+4}, \alpha_{3a+5}, \alpha_{3a+6}$ are located on the left side of lower twists from the left. Figure 5 exhibits this numbering convention when $a = 1$.

Figure 6 shows the convention of sign for each crossing. The $(3a+6) \times (3a+6)$ Goeritz matrix G is defined as follows. For $i \neq j$, the (i, j) -entry of G is the sum of signs at all the crossings between the regions α_i and α_j . The (i, i) -entry is $-\sum \text{sign}(c)$, where the sum is over all crossings c around the region α_i . Then it is well known that $|\det G|$ equals to the determinant of \mathcal{L} .

For example, if $a = 1$, then the Goeritz matrix G_1 is

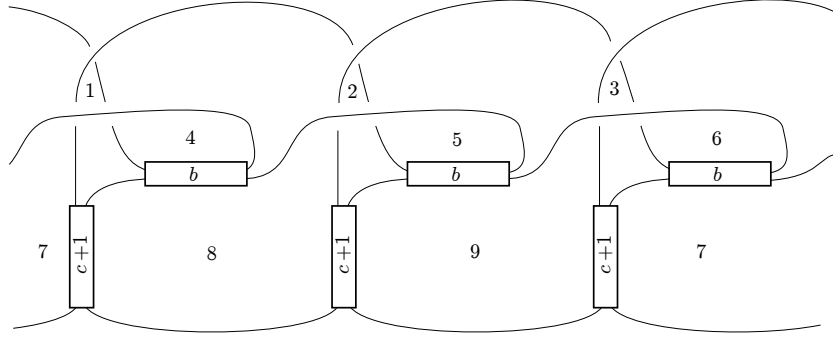
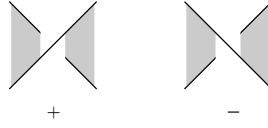
FIGURE 5. The link \mathcal{L} and the white regions when $a = 1$ 

FIGURE 6. Signs of crossing

$$\left(\begin{array}{c|cc|ccc} -I & & -I & & & & I \\ \hline & & & & 0 & -b & 0 \\ -I & & (b+1)I & & 0 & 0 & -b \\ & & & & -b & 0 & 0 \\ \hline & 0 & 0 & -b & b+2c+1 & -c-1 & -c-1 \\ I & -b & 0 & 0 & -c-1 & b+2c+1 & -c-1 \\ & 0 & -b & 0 & -c-1 & -c-1 & b+2c+1 \end{array} \right),$$

where I denotes the 3×3 identity matrix. Then a direct calculation shows $\det G_1 = (3bc + 6b + 6c + 5)^2$. Since this value is positive, we have $\det \mathcal{L} = \det G_1$.

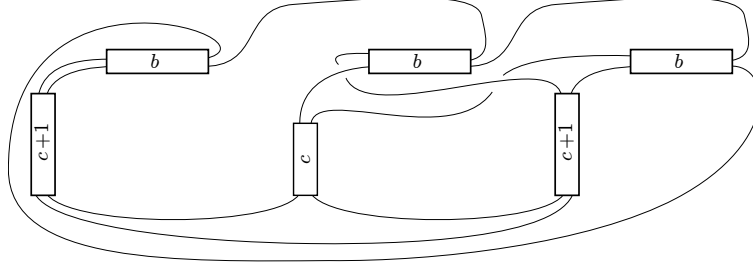
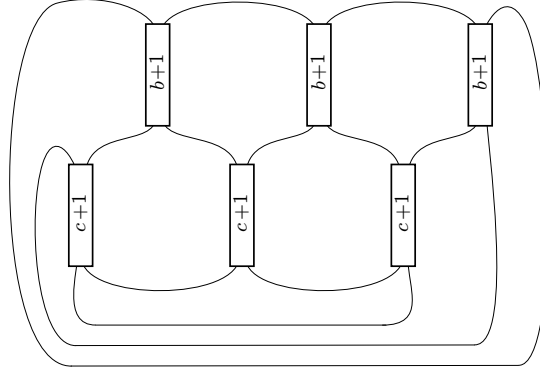
4. THE CASE WHERE $a = 1$

The purpose of this section is to show that the link \mathcal{L} is quasi-alternating when $a = 1$. The link diagram D is illustrated in Figure 5. For $i \in \{1, 2, 3\}$, let c_i be the upper crossing of the white region α_i . Let $\varepsilon_i \in \{*, \infty, 0\}$. We use the notation $L(\varepsilon_1, \varepsilon_2, \varepsilon_3)$ to express the link obtained from the link diagram D by performing a resolution of type ε_i at the crossing c_i . Here, if $\varepsilon_i = *$, then the crossing c_i is not changed. If $\varepsilon_i = \infty$ or 0 , then c_i is split vertically or horizontally, respectively, as in Figure 4.

Lemma 4.1. (1) $L(0, 0, *) = L(0, \infty, 0) = P(b + c + 1, b + c + 1, b + c + 1)$.

Hence these are alternating.

(2) $L(0, \infty, \infty) = L(\infty, 0, \infty) = L(\infty, \infty, 0)$, and these are alternating.

FIGURE 7. $L(0, \infty, \infty)$ is alternatingFIGURE 8. $L(\infty, \infty, \infty)$

(3) $L(\infty, \infty, \infty)$ is quasi-alternating.

Proof. (1) This is obvious from their diagrams.

(2) The equivalence of three links follows from the symmetry. An alternating diagram of $L(0, \infty, \infty)$ is illustrated in Figure 7.

(3) The link $L(\infty, \infty, \infty)$ is equivalent to one as in Figure 8. This link is shown to be quasi-alternating by Peters [13]. \square

To conclude that \mathcal{L} is quasi-alternating, we need the values of determinants of some of links $L(\varepsilon_1, \varepsilon_2, \varepsilon_3)$.

Recall that the diagram D (Figure 5) of \mathcal{L} yields the Goeritz matrix G_1 described in Section 3. For $L(0, *, *)$ (resp. $L(\infty, *, *)$), its diagram is obtained from D by splitting the crossing c_1 horizontally (resp. vertically). Then, the corresponding Goeritz matrix is obtained from G_1 by replacing the $(1, 1)$ -entry with 0, or deleting the first row and column, respectively. In this way, calculating determinants of the matrices gives Table 1.

Theorem 4.2. *Assume $a = 1$. Then the link \mathcal{L} is quasi-alternating. Furthermore, $L(0, *, *)$ and $L(\infty, *, *)$ are quasi-alternating.*

Link	Determinant
$L(0, *, *)$	$2(b + c + 1)(3bc + 6b + 6c + 5)$
$L(\infty, *, *)$	$(3bc + 4b + 4c + 3)(3bc + 6b + 6c + 5)$
$L(0, 0, *)$	$3(b + c + 1)^2$
$L(0, \infty, *)$	$(b + c + 1)(6bc + 9b + 9c + 7)$
$L(\infty, 0, *)$	$(b + c + 1)(6bc + 9b + 9c + 7)$
$L(\infty, \infty, *)$	$(3bc + 3b + 3c + 2)(3bc + 5b + 5c + 4)$
$L(0, \infty, 0)$	$3(b + c + 1)^2$
$L(0, \infty, \infty)$	$2(b + c + 1)(3bc + 3b + 3c + 2)$
$L(\infty, \infty, \infty)$	$(3bc + 3b + 3c + 2)^2$

TABLE 1. Determinants of links

Proof. By Lemma 4.1, $L(0, \infty, 0)$ and $L(0, \infty, \infty)$ are alternating. As shown in Table 1, we have $\det L(0, \infty, *) = \det L(0, \infty, 0) + \det L(0, \infty, \infty)$. Hence $L(0, \infty, *)$ is quasi-alternating. Similarly, because $L(0, 0, *)$ is alternating and $\det L(0, *, *) = \det L(0, 0, *) + \det L(0, \infty, *)$, $L(0, *, *)$ is quasi-alternating. Also, we can verify that $L(\infty, *, *)$ is quasi-alternating by the same argument. Finally, the equation $\det \mathcal{L} = \det L(0, *, *) + \det L(\infty, *, *)$ implies the conclusion that \mathcal{L} is quasi-alternating. \square

5. INDUCTION

As in Section 4, we use the notation $L(a: \varepsilon_1, \varepsilon_2, \varepsilon_3)$ with $\varepsilon \in \{*, \infty, 0\}$ to denote the link obtained from \mathcal{L} by performing the resolution of type ε_i at the crossing c_i . Here, c_i is located at the top of the white region α_i . See Figure 3. Because we will use an inductive argument, the parameter a is added. In particular, $\mathcal{L} = L(a: *, *, *)$.

Lemma 5.1. *Suppose $a > 1$.*

- (1) $L(a: 0, 0, *) = L(a: 0, \infty, 0) = L(a: \infty, 0, 0) = P(b+c+1, b+c+1, b+c+1)$,
and these are alternating.
- (2) $L(a: 0, \infty, \infty) = L(a: \infty, 0, \infty) = L(a: \infty, \infty, 0) = L(a-1: 0, *, *)$.
- (3) $L(a: \infty, \infty, \infty) = L(a-1: *, *, *)$.

Proof. These immediately follow from the diagrams. \square

Lemma 5.2. *For \mathcal{L} , $L(a: 0, *, *)$ and $L(a: \infty, *, *)$,*

$$\begin{aligned}
\det \mathcal{L} &= (3ab + 3bc + 3ca + 3a + 3b + 3c + 2)^2, \\
\det L(a: 0, *, *) &= 2(b + c + 1)(3ab + 3bc + 3ca + 3a + 3b + 3c + 2), \\
\det L(a: \infty, *, *) &= (3ab + 3bc + 3ca + 3a + b + c)(3ab + 3bc + 3ca + 3a + 3b + 3c + 2).
\end{aligned}$$

Hence $\det \mathcal{L} = \det L(a: 0, *, *) + \det L(a: \infty, *, *)$.

Proof. Let G be the $(3a + 6) \times (3a + 6)$ Goeritz matrix obtained from the link diagram D of \mathcal{L} . As in Section 3,

$$G = \left(\begin{array}{ccc|ccc} -2I & I & & & & \\ & I & \ddots & & & \\ & & & -2I & I & \\ & & & I & -2I & \\ \hline & & & I & & \\ & & & O & & \\ & & & O & & \end{array} \middle| \begin{array}{ccc} & & \\ I & O & O \\ \hline & G_1 & \end{array} \right),$$

where I is the 3×3 identity matrix, O is the 3×3 zero matrix, and G_1 is exactly the 9×9 matrix given in Section 3. To calculate its determinant, add the i -th column multiplied by $1/2$ to the $(i + 3)$ -th column for $i = 1, 2, 3$. Then reduce the matrix to a $(3a + 3) \times (3a + 3)$ matrix. By repeating this process, we have $\det G = (-1)^{a-1} a^3 \det G'_1$, where G'_1 is obtained from G_1 by replacing the upper left 3×3 block $-I$ with $-\frac{1}{a}I$. Thus $\det G = (-1)^{a-1} (3ab + 3bc + 3ca + 3a + 3b + 3c + 2)^2$, and so $\det \mathcal{L} = (3ab + 3bc + 3ca + 3a + 3b + 3c + 2)^2$.

Consider the diagram of $L(a: 0, *, *)$ obtained from the diagram D (Figure 3) by splitting the crossing c_1 horizontally. The corresponding Goeritz matrix G_0 is the above G with replacing the $(1, 1)$ -entry with -1 . Add the first column to the 4-th column, and the i -th column multiplied by $1/2$ to the $(i + 3)$ -th column for $i = 2, 3$. Then reduce the size as before. Repeating this gives $\det G_0 = (-1)^{a-1} a^2 \det G''_1$, where G''_1 is obtained from G_1 by replacing the (i, i) -entry with $0, -1/a, -1/a$, respectively, for $i = 1, 2, 3$. Then we have $\det L(a: 0, *, *) = 2(b + c + 1)(3ab + 3bc + 3ca + 3a + 3b + 3c + 2)$.

Finally, the diagram of $L(a: \infty, *, *)$ is obtained from D by splitting the crossing c_1 vertically. The corresponding Goeritz matrix G_∞ is G with deleting the first column and row. Add the i -th column multiplied by $1/2$ to the $(i + 3)$ -th column for $i = 1, 2$. Then reduce the size of matrix. Repeating this yields $\det G_\infty = (-1)^{a-3} (a-1) a^2 \det G'''_1$, where G'''_1 is obtained from G_1 by replacing the (i, i) -entry with $-1/(a-1), -1/a, -1/a$, respectively, for $i = 1, 2, 3$. Then $\det L(a: \infty, *, *) = (3ab + 3bc + 3ca + 3a + b + c)(3ab + 3bc + 3ca + 3a + 3b + 3c + 2)$. \square

By a similar process to the proof of Lemma 5.2, we can calculate of determinants of some other links, as in Table 2. We omit the details.

Link	Determinant
$L(a: 0, 0, *)$	$3(b + c + 1)^2$
$L(a: 0, \infty, *)$	$(b + c + 1)(6ab + 6bc + 6ca + 6a + 3b + 3c + 1)$
$L(a: \infty, \infty, *)$	$(3ab + 3bc + 3ca + 3a + 2b + 2c + 1)(3ab + 3bc + 3ca + 3a - 1)$

TABLE 2. Determinants of links

Lemma 5.3. *We have the following equations.*

$$\begin{aligned}
\det L(a: 0, *, *) &= \det L(a: 0, 0, *) + \det L(a: 0, \infty, *), \\
\det L(a: 0, \infty, *) &= \det L(a: 0, \infty, 0) + \det L(a: 0, \infty, \infty), \\
\det L(a: \infty, *, *) &= \det L(a: \infty, 0, *) + \det L(a: \infty, \infty, *), \\
\det L(a: \infty, 0, *) &= \det L(a: \infty, 0, 0) + \det L(a: \infty, 0, \infty), \\
\det L(a: \infty, \infty, *) &= \det L(a: \infty, \infty, 0) + \det L(a: \infty, \infty, \infty).
\end{aligned}$$

Proof. These immediately follow from Lemmas 5.1, 5.2 and Table 2. \square

Proof of Theorem 2.2. We prove a stronger claim that not only $\mathcal{L} = L(a: *, *, *)$ but $L(a: 0, *, *)$ is quasi-alternating. The proof is done by induction on a . By Theorem 4.2, the claim is true when $a = 1$. Suppose $a > 1$ and that the claim holds for $a - 1$.

By Lemma 5.2, if both $L(a: 0, *, *)$ and $L(a: \infty, *, *)$ are quasi-alternating, then \mathcal{L} is quasi-alternating.

First, consider $L(a: 0, *, *)$. By the resolution at the crossing c_2 , we obtain $L(a: 0, 0, *)$ and $L(a: 0, \infty, *)$. For the latter, perform the resolution at the crossing c_3 to yield $L(a: 0, \infty, 0)$ and $L(a: 0, \infty, \infty)$. Then the claim that $L(a: 0, *, *)$ is quasi-alternating follows from the facts that $L(a: 0, 0, *)$ and $L(a: 0, \infty, 0)$ are alternating (Lemma 5.1) and $L(a: 0, \infty, \infty) (= L(a - 1: 0, *, *))$ is quasi-alternating by our inductive assumption, coupled with the equations among determinants (Lemma 5.3). Similarly, we can show that $L(a: \infty, *, *)$ is quasi-alternating. \square

6. REMARKS

(1) Boyer, Gordon and Watson [2] propose a conjecture that an irreducible rational homology 3-sphere is an L -space if and only if its fundamental group is not left-orderable. For $K = P(2a + 1, 2b + 1, 2c + 1)$, $\pi_1 \Sigma_2(K)$ is not left-orderable, since $\Sigma_2(K)$ is a Seifert-fibered L -space ([2]). By Theorem 1.1, $\Sigma_3(K)$ is also an L -space. Hence it is an interesting task to show that $\pi_1 \Sigma_3(K)$ is not left-orderable.

(2) Among genus one pretzel knots, for example, $P(-3, 5, 5)$ is non-alternating. It is known that its double branched cover is not an L -space ([5, 7]). Thus we may not expect that the 3-fold cyclic branched cover is an L -space.

(3) Let K be a pretzel knot $P(3, 3, -n)$ with $n \geq 3$, odd. If $n > 3$, then K is quasi-alternating, but $P(3, 3, -3)$, which is 9_{46} in the knot table, is not quasi-alternating (see [5, 7]). Nevertheless, $\Sigma_2(K)$ is always an L -space. By a similar argument, we can show that $\Sigma_3(K)$ is an L -space, but the details will be treated elsewhere.

(4) For an alternating pretzel knot $P(2a+1, 2b+1, 2c+1)$, we may expect that the d -fold cyclic branched cover is an L -space for at least small $d \geq 4$.

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